

Gradient estimates for the porous medium equations on Riemannian manifolds*

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Abstract. In this paper we study gradient estimates for the positive solutions of the porous medium equation:

$$u_t = \Delta u^m$$

where $m > 1$, which is a nonlinear version of the heat equation. We derive local gradient estimates of the Li-Yau type for positive solutions of porous medium equations on Riemannian manifolds with Ricci curvature bounded from below. As applications, several parabolic Harnack inequalities are obtained. In particular, our results improve the ones of Lu, Ni, Vázquez and Villani in [10]. Moreover, our results recover the ones of Davies in [4], Hamilton in [5] and Li and Xu in [7].

Keywords: Porous medium equation, Li-Yau type estimate, Harnack inequality

Mathematics Subject Classification: Primary 35B45, Secondary 35K55

1 Introduction

Let (M^n, g) be an n -dimensional complete Riemannian manifold. Li and Yau [8] studied positive solutions of the heat equation

$$u_t = \Delta u \tag{1.1}$$

and obtained the following gradient estimates:

Theorem A(Li-Yau [8]). *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution of (1.1) on $B_p(2R) \times [0, T]$. Then on $B_p(R)$, we have*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{C\alpha^2}{R^2} \left(\frac{\alpha^2}{\alpha - 1} + \sqrt{KR} \right) + \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}, \tag{1.2}$$

where $\alpha > 1$ is a constant and the constant C depends only on n . Moreover, when $R \rightarrow \infty$, (1.2) yields the following estimate on complete noncompact Riemannian manifold (M^n, g) :

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}. \tag{1.3}$$

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In [4], Davies improved the estimate (1.3) to

Theorem B(Davies [4]). *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.1). Then*

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{4(\alpha - 1)} + \frac{n\alpha^2}{2t}. \quad (1.4)$$

In [5], Hamilton proved the following estimate:

Theorem C(Hamilton [5]). *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.1). Then*

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{u_t}{u} \leq e^{4Kt} \frac{n}{2t}. \quad (1.5)$$

Recently, Li and Xu [7] obtained new Li-Yau type gradient estimates for positive solutions of the heat equation (1.1) on Riemannian manifolds. For the related research and improvement in this direction, see [2, 3, 5, 6, 9, 11, 14–16] and the references therein.

The porous medium equation

$$u_t = \Delta u^m, \quad (1.6)$$

where $m > 1$ is a nonlinear version of the heat equation (1.1). For various values of $m > 1$, it has arisen in different applications to model diffusive phenomena. The readers who are interested in the applications of (1.6) see [1, 10, 13] and the references therein. In [10], Lu, Ni, Vázquez and Villani studied gradient estimates of (1.6) with $m > 1$ and proved the following results (see Theorem 3.3 in [10]):

Theorem D(P. Lu, L. Ni, J. Vázquez, C. Villani [10]). *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \max_{B_p(2R) \times [0, T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} &\leq \frac{C M a \alpha^2}{R^2} \left(\frac{\alpha^2}{\alpha - 1} a m^2 + (m - 1)(1 + \sqrt{K}R) \right) \\ &\quad + \frac{\alpha^2}{\alpha - 1} a(m - 1) M K + \frac{a \alpha^2}{t}, \end{aligned} \quad (1.7)$$

where $a = \frac{n(m-1)}{n(m-1)+2}$ and the constant C depends only on n . Moreover, when $R \rightarrow \infty$, (1.7) yields the following estimate on complete noncompact Riemannian manifold (M^n, g) :

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{\alpha - 1} a(m - 1) M K + \frac{a \alpha^2}{t}. \quad (1.8)$$

In this paper, we further study gradient estimates of the porous medium equation (1.6). We derive Davies's type estimate and Hamilton's to (1.6). Besides, we obtain estimates of Li-Xu type for (1.6). In particular, our results improve the ones of Lu, Ni, Vázquez and Villani in [10]. Now, we state our results as follows:

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \max_{B_p(2R) \times [0, T]} v$. Then for any $\alpha > 1$, on the ball $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq & \left\{ \frac{a\alpha^2 m M^{\frac{1}{2}} C}{(\alpha-1)^{\frac{1}{2}} R} + a^{\frac{1}{2}} \alpha \left[\frac{1}{t} + \frac{(m-1)MK}{2(\alpha-1)} \right. \right. \\ & \left. \left. + (m-1)M \frac{C}{R^2} \left(1 + \sqrt{K} \coth(\sqrt{K}R) \right) \right]^{\frac{1}{2}} \right\}^2, \end{aligned} \quad (1.9)$$

where $a = \frac{n(m-1)}{n(m-1)+2}$ and the constant C depends only on n .

Letting $R \rightarrow \infty$, we obtain the gradient estimates on complete noncompact Riemannian manifolds, which improves (1.8) of Theorem D in [10].

Corollary 1.1. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$. Then for any $\alpha > 1$, we have*

$$\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \leq \frac{\alpha^2}{2(\alpha-1)} a(m-1)MK + \frac{a\alpha^2}{t}. \quad (1.10)$$

Applying the inequality (1.10), we derive the following Harnack inequality:

Corollary 1.2. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$, $\tilde{M} = \inf_{M^n \times [0, T]} v$. Then for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, the following inequality holds:*

$$\begin{aligned} v(x_1, t_1) \leq & v(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{a\alpha} \exp \left\{ \frac{\alpha \text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} \right. \\ & \left. + \frac{\alpha}{2(\alpha-1)} a(m-1)MK(t_2 - t_1) \right\}, \end{aligned} \quad (1.11)$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 1.1. We rewrite the inequality (1.8) as

$$|\nabla v|^2 - \alpha v_t \leq \frac{\alpha^2}{\alpha-1} a(m-1)MKv + \frac{a\alpha^2 v}{t}. \quad (1.12)$$

Since $(m-1)v = mu^{m-1}$, we have $(m-1)v \rightarrow 1$ as $m \rightarrow 1$. Hence, $(m-1)M \rightarrow 1$,

$$\begin{aligned} |\nabla v|^2 & \rightarrow \frac{|\nabla u|^2}{u^2}, \\ v_t & \rightarrow \frac{u_t}{u}, \\ av & \rightarrow \frac{n}{2}, \end{aligned}$$

as $m \rightarrow 1$. As a result, (1.12) becomes the inequality (1.3) in Theorem A of Li-Yau. Therefore, for complete noncompact Riemannian manifold (M^n, g) , the estimate (1.8) in

Theorem D of Lu, Ni, Vázquez and Villani reduces to the estimate (1.3) in Theorem A of Li-Yau when $m \rightarrow 1$. Similarly, it is easy to see that our estimates (1.10) reduces to the estimate (1.4) in Theorem B of Davies when $m \rightarrow 1$.

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \max_{B_p(2R) \times [0, T]} v$. Then on the ball $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} &\leq \left(\frac{m^2 M a^2 \alpha^4(t)}{2(\alpha(t) - 1)} + 3(m-1) M a \alpha^2(t) \right) \frac{C}{R^2} \\ &\quad + (m-1) M a \alpha^2(t) \sqrt{K} \coth(\sqrt{K} R) \frac{C}{R} + \frac{a \alpha^2(t)}{t}, \end{aligned} \quad (1.13)$$

where $a = \frac{n(m-1)}{n(m-1)+2}$, $\alpha(t) = e^{2(m-1)MKt}$ and the constant C depends only on n .

Letting $R \rightarrow \infty$, we obtain the following gradient estimates on complete noncompact Riemannian manifolds.

Corollary 1.3. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$. Then we have*

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq \frac{a \alpha^2(t)}{t}, \quad (1.14)$$

where $\alpha(t) = e^{2(m-1)MKt}$.

Applying the inequality (1.14), we derive the following Harnack inequality:

Corollary 1.4. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$, $\tilde{M} = \inf_{M^n \times [0, T]} v$. Then for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, $\alpha > 1$, the following inequality holds:*

$$v(x_1, t_1) \leq v(x_2, t_2) \exp \left\{ \frac{e^{2(m-1)MKt_2} - e^{2(m-1)MKt_1}}{2(m-1)MK} \left(\frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)^2} + \frac{a}{t_1} \right) \right\}, \quad (1.15)$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 1.2. Notice that (1.14) can be written as

$$|\nabla v|^2 - \alpha(t) v_t \leq \frac{av}{t} \alpha^2(t). \quad (1.16)$$

Since $(m-1)v = mu^{m-1}$, we have $(m-1)v \rightarrow 1$ as $m \rightarrow 1$. Hence, $(m-1)M \rightarrow 1$, $av \rightarrow \frac{n}{2}$ and $\alpha(t) \rightarrow e^{2Kt}$. Hence letting $m \rightarrow 1$ in (1.16) yields the inequality (1.5). Therefore, our Corollary 1.3 extends Theorem C of Hamilton.

Theorem 1.3. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \max_{B_p(2R) \times [0, T]} v$. Then on the ball $B_p(R)$, we have*

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) &\leq \left\{ a(m-1) \left(\frac{C}{R^2} + \frac{C\sqrt{K} \coth(\sqrt{K} R)}{R} \right) \right. \\ &\quad \left. + a^2 m^2 \frac{C}{R^2 \tanh((m-1)MKt)} \right\} M, \end{aligned} \quad (1.17)$$

where $a = \frac{n(m-1)}{n(m-1)+2}$ and the constant C depends only on n . $\alpha(t)$ and $\varphi(t)$ are given by

$$\begin{aligned}\varphi(t) &= a(m-1)MK\{\coth((m-1)MKt) + 1\}, \\ \alpha(t) &= 1 + \frac{\cosh((m-1)MKt) \sinh((m-1)MKt) - (m-1)MKt}{\sinh^2((m-1)MKt)}.\end{aligned}\quad (1.18)$$

Letting $R \rightarrow \infty$, we obtain the gradient estimates on complete noncompact Riemannian manifolds.

Corollary 1.5. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$. Then we have*

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0, \quad (1.19)$$

where $\alpha(t)$ and $\varphi(t)$ are given by (1.18).

Applying the inequality (1.19), we derive the following Harnack inequality:

Corollary 1.6. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$, $\tilde{M} = \inf_{M^n \times [0, T]} v$. Then for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, the following inequality holds:*

$$v(x_1, t_1) \leq v(x_2, t_2) A_1(t_1, t_2) \exp \left\{ \frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} (1 + A_2(t_1, t_2)) \right\}, \quad (1.20)$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 . Moreover,

$$\begin{aligned}A_1(t_1, t_2) &= \left(\frac{\exp(2(m-1)MKt_2) - 2(m-1)MKt_2 - 1}{\exp(2(m-1)MKt_1) - 2(m-1)MKt_1 - 1} \right)^{\frac{a}{2}}, \\ A_2(t_1, t_2) &= \frac{t_2 \coth((m-1)MKt_2) - t_1 \coth((m-1)MKt_1)}{t_2 - t_1}.\end{aligned}$$

A linear version of Theorem 1.3 is the following:

Theorem 1.4. *Let (M^n, g) be a complete Riemannian manifold with $\text{Ric}(B_p(2R)) \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \max_{B_p(2R) \times [0, T]} v$. Then on the ball $B_p(R)$, we have*

$$\begin{aligned}\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) &\leq \left\{ a(m-1)\alpha^2(t) \left(\frac{C}{R^2} + \frac{C\sqrt{K} \coth(\sqrt{K}R)}{R} \right) \right. \\ &\quad \left. + \frac{a^2 m^2 \alpha^4(t)}{\beta(t)} \frac{C}{R^2} \right\} M,\end{aligned}\quad (1.21)$$

where $a = \frac{n(m-1)}{n(m-1)+2}$ and the constant C depends only on n . $\alpha(t)$ and $\varphi(t)$ are given by

$$\begin{aligned}\varphi(t) &= \frac{a}{t} + a(m-1)MK + \frac{a}{3}((m-1)MK)^2 t, \\ \alpha(t) &= 1 + \frac{2}{3}(m-1)MKt.\end{aligned}\quad (1.22)$$

Letting $R \rightarrow \infty$, we obtain the gradient estimates on complete noncompact Riemannian manifolds.

Corollary 1.7. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$. Then we have*

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0, \quad (1.23)$$

where $\alpha(t)$ and $\varphi(t)$ are given by (1.22).

Applying the inequality (1.23), we derive the following Harnack inequality:

Corollary 1.8. *Let (M^n, g) be a complete noncompact Riemannian manifold with $\text{Ric} \geq -K$, $K \geq 0$. Suppose that u is a positive solution to (1.6). Let $v = \frac{m}{m-1}u^{m-1}$ and $M = \sup_{M^n \times [0, T]} v$, $\tilde{M} = \inf_{M^n \times [0, T]} v$. Then for any $x_1, x_2 \in M^n$, $0 < t_1 < t_2 < T$, the following inequality holds:*

$$\begin{aligned} v(x_1, t_1) \leq & v(x_2, t_2) \left(\frac{t_2}{t_1} \right)^a \left(\frac{1 + \frac{2}{3}(m-1)MKt_2}{1 + \frac{2}{3}(m-1)MKt_1} \right)^{\frac{-a}{4}} \\ & \exp \left\{ \frac{\text{dist}^2(x_2, x_1)}{4\tilde{M}(t_2 - t_1)} \left(1 + \frac{1}{3}(m-1)MK(t_2 + t_1) \right) + \frac{a}{2}(m-1)MK(t_2 - t_1) \right\}, \end{aligned} \quad (1.24)$$

where $\text{dist}(x_2, x_1)$ is the distance between x_1 and x_2 .

Remark 1.3. When $m \rightarrow 1$, our Theorem 1.3 reduces to Theorem 1.1 of Li and Xu in [7]. Similarly, our Theorem 1.4 reduces to Theorem 1.2 of Li and Xu in [7]. Note that (1.17) can be written as

$$\begin{aligned} |\nabla v|^2 - \alpha(t)v_t - \varphi(t)v \leq & \left\{ a(m-1) \left(\frac{C}{R^2} + \frac{C\sqrt{K} \coth(\sqrt{K}R)}{R} \right) \right. \\ & \left. + a^2m^2 \frac{C}{R^2 \tanh((m-1)MKt)} \right\} Mv. \end{aligned} \quad (1.25)$$

Since $(m-1)v = mu^{m-1}$, we have $(m-1)v \rightarrow 1$ as $m \rightarrow 1$. Hence, $(m-1)M \rightarrow 1$,

$$\begin{aligned} \alpha(t) & \rightarrow 1 + \frac{\cosh(Kt) \sinh(Kt) - Kt}{\sinh^2(Kt)}, \\ \varphi(t)v & \rightarrow \frac{nK}{2} \{ \coth(Kt) + 1 \}, \\ |\nabla v|^2 & \rightarrow \frac{|\nabla u|^2}{u^2}, \\ v_t & \rightarrow \frac{u_t}{u}, \end{aligned}$$

and

$$\begin{aligned} & \left\{ a(m-1) \left(\frac{C}{R^2} + \frac{C\sqrt{K} \coth(\sqrt{K}R)}{R} \right) \right. \\ & \left. + a^2m^2 \frac{C}{R^2 \tanh((m-1)MKt)} \right\} Mv \rightarrow \frac{nC}{R^2} + \frac{nC\sqrt{K}}{R} \coth(\sqrt{K}R) + \frac{n^2C}{R^2 \tanh(Kt)} \end{aligned}$$

as $m \rightarrow 1$. As a result, (1.25) becomes

$$\frac{|\nabla u|^2}{u^2} - \tilde{\alpha}(t) \frac{u_t}{u} - \tilde{\varphi}(t) \leq \frac{nC}{R^2} + \frac{nC\sqrt{K}}{R} \coth(\sqrt{K}R) + \frac{n^2C}{R^2 \tanh(Kt)} \quad (1.26)$$

by letting $m \rightarrow 1$, where $\tilde{\alpha}(t), \tilde{\varphi}(t)$ in (1.26) are given by $\tilde{\alpha}(t) = 1 + \frac{\cosh(Kt) \sinh(Kt) - Kt}{\sinh^2(Kt)}$, $\tilde{\varphi}(t) = \frac{nK}{2} \{\coth(Kt) + 1\}$. Therefore, our Theorem 1.3 becomes Theorem 1.1 of Li and Xu in [7] as long as letting $m \rightarrow 1$. Similarly, our Theorem 1.4 becomes Theorem 1.2 of Li and Xu in [7] as long as letting $m \rightarrow 1$.

Remark 1.4. When t is small enough, $\alpha(t), \varphi(t)$ defined by (1.18) and (1.22) both satisfy $\alpha(t) \rightarrow 1$ and $\varphi(t) \leq 2a(m-1)MK + \frac{a}{t}$. Hence, by Corollary 1.5 and 1.7, we have

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} \leq 2a(m-1)MK + \frac{a}{t}. \quad (1.27)$$

Clearly, for t small enough, (1.27) is better than (1.8). Thus Corollary 1.5 and 1.7 improve (1.8) in Theorem D of [10] in this sense.

2 Proof of Theorem 1.1

Let $v = \frac{m}{m-1}u^{m-1}$. From the equation (1.6), one gets $v_t = (m-1)v\Delta v + |\nabla v|^2$ which is equivalent to the following form:

$$\frac{v_t}{v} = (m-1)\Delta v + \frac{|\nabla v|^2}{v}. \quad (2.1)$$

Lemma 2.1. *As in [10], we introduce the differential operator*

$$\mathcal{L} = \partial_t - (m-1)v\Delta.$$

Denote by $F = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \varphi$, where $\alpha = \alpha(t)$ and $\varphi = \varphi(t)$ are functions depending on t , then we have

$$\begin{aligned} \mathcal{L}(F) = & -2(m-1)v_{ij}^2 - 2(m-1)R_{ij}v_iv_j + 2m\nabla v \nabla F \\ & - ((m-1)\Delta v)^2 + (1-\alpha) \left(\frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} - \varphi'. \end{aligned} \quad (2.2)$$

Proof of Lemma 2.1. We need two formulas (see p5-p6 in [10])

$$\mathcal{L} \left(\frac{v_t}{v} \right) = (m-1)\Delta v \frac{v_t}{v} + \frac{2}{v} \nabla v \nabla v_t - \frac{v_t}{v} \frac{|\nabla v|^2}{v} + 2(m-1)v \nabla \left(\frac{v_t}{v} \right) \nabla (\log v),$$

$$\begin{aligned} \mathcal{L} \left(\frac{|\nabla v|^2}{v} \right) = & 2(m-1)\Delta v \frac{|\nabla v|^2}{v} + \frac{2}{v} \nabla v \nabla |\nabla v|^2 - 2(m-1)v_{ij}^2 \\ & - 2(m-1)R_{ij}v_iv_j - \frac{|\nabla v|^4}{v^2} + 2(m-1)v \nabla \left(\frac{|\nabla v|^2}{v} \right) \nabla (\log v). \end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{L}(F) &= \mathcal{L}\left(\frac{|\nabla v|^2}{v}\right) - \alpha \mathcal{L}\left(\frac{v_t}{v}\right) - \alpha' \frac{v_t}{v} - \varphi' \\
&= 2(m-1)\Delta v \frac{|\nabla v|^2}{v} + \frac{2}{v} \nabla v \nabla |\nabla v|^2 - 2(m-1)v_{ij}^2 \\
&\quad - 2(m-1)R_{ij}v_i v_j - \frac{|\nabla v|^4}{v^2} + 2(m-1)v \nabla \left(\frac{|\nabla v|^2}{v}\right) \nabla(\log v) \\
&\quad - \alpha(m-1)\Delta v \frac{v_t}{v} - \alpha \frac{2}{v} \nabla v \nabla v_t + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} - 2\alpha(m-1)v \nabla \left(\frac{v_t}{v}\right) \nabla(\log v) \\
&\quad - \alpha' \frac{v_t}{v} - \varphi'.
\end{aligned} \tag{2.3}$$

Since

$$\begin{aligned}
2(m-1)v \nabla \left(\frac{|\nabla v|^2}{v}\right) \nabla(\log v) - 2\alpha(m-1)v \nabla \left(\frac{v_t}{v}\right) \nabla(\log v) &= 2(m-1)\nabla v \nabla F, \\
\frac{2}{v} \nabla v \nabla |\nabla v|^2 - \alpha \frac{2}{v} \nabla v \nabla v_t &= \frac{2}{v} \nabla v \nabla ((F + \varphi)v) = 2(F + \varphi) \frac{|\nabla v|^2}{v} + 2\nabla v \nabla F,
\end{aligned}$$

we have

$$\begin{aligned}
2(m-1)v \nabla \left(\frac{|\nabla v|^2}{v}\right) \nabla(\log v) - 2\alpha(m-1)v \nabla \left(\frac{v_t}{v}\right) \nabla(\log v) &+ \frac{2}{v} \nabla v \nabla |\nabla v|^2 - \alpha \frac{2}{v} \nabla v \nabla v_t \\
&= 2m \nabla v \nabla F + 2(F + \varphi) \frac{|\nabla v|^2}{v} \\
&= 2m \nabla v \nabla F + 2 \left(\frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \right) \frac{|\nabla v|^2}{v}.
\end{aligned} \tag{2.4}$$

On the other hand, it follows from (2.1) that

$$\begin{aligned}
2(m-1)\Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha(m-1)\Delta v \frac{v_t}{v} + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \\
&= 2 \frac{|\nabla v|^2}{v} \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) - \frac{|\nabla v|^4}{v^2} - \alpha \frac{v_t}{v} \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \\
&= (2\alpha + 2) \frac{v_t}{v} \frac{|\nabla v|^2}{v} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left(\frac{v_t}{v} \right)^2.
\end{aligned} \tag{2.5}$$

Therefore, (2.4) and (2.5) give

$$\begin{aligned}
2(m-1)v \nabla \left(\frac{|\nabla v|^2}{v}\right) \nabla(\log v) - 2\alpha(m-1)v \nabla \left(\frac{v_t}{v}\right) \nabla(\log v) &+ \frac{2}{v} \nabla v \nabla |\nabla v|^2 \\
&- \alpha \frac{2}{v} \nabla v \nabla v_t + 2(m-1)\Delta v \frac{|\nabla v|^2}{v} - \frac{|\nabla v|^4}{v^2} - \alpha(m-1)\Delta v \frac{v_t}{v} + \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \\
&= 2m \nabla v \nabla F - \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1 - \alpha) \left(\frac{v_t}{v} \right)^2 \\
&= 2m \nabla v \nabla F - ((m-1)\Delta v)^2 + (1 - \alpha) \left(\frac{v_t}{v} \right)^2.
\end{aligned} \tag{2.6}$$

Putting (2.6) into (2.3) yields

$$\begin{aligned}\mathcal{L}(F) = & -2(m-1)v_{ij}^2 - 2(m-1)R_{ij}v_iv_j + 2m\nabla v\nabla F \\ & - ((m-1)\Delta v)^2 + (1-\alpha)\left(\frac{v_t}{v}\right)^2 - \alpha'\frac{v_t}{v} - \varphi'.\end{aligned}$$

It completes the proof of Lemma 2.1.

Now we prove Theorem 1.1. Define $\tilde{F} = \frac{|\nabla v|^2}{v} - \alpha\frac{v_t}{v}$, where $\alpha > 1$ is a constant. Then we have from (2.2)

$$\mathcal{L}(\tilde{F}) = -2(m-1)v_{ij}^2 - 2(m-1)R_{ij}v_iv_j + 2m\nabla v\nabla\tilde{F} - ((m-1)\Delta v)^2 + (1-\alpha)\left(\frac{v_t}{v}\right)^2.$$

Under the assumption that $\text{Ric} \geq -K$ and the definition of M , we have

$$\begin{aligned}\mathcal{L}(\tilde{F}) = & -2(m-1)v_{ij}^2 - 2(m-1)R_{ij}v_iv_j + 2m\nabla v\nabla\tilde{F} - ((m-1)\Delta v)^2 + (1-\alpha)\left(\frac{v_t}{v}\right)^2 \\ \leq & -\frac{2}{n(m-1)}((m-1)\Delta v)^2 + 2(m-1)K|\nabla v|^2 + 2m\nabla v\nabla\tilde{F} - ((m-1)\Delta v)^2 \\ \leq & -\frac{1}{a}((m-1)\Delta v)^2 + 2(m-1)MK\frac{|\nabla v|^2}{v} + 2m\nabla v\nabla\tilde{F} \\ = & -\frac{1}{a\alpha^2}\left(\tilde{F} + (\alpha-1)\frac{|\nabla v|^2}{v}\right)^2 + 2(m-1)MK\frac{|\nabla v|^2}{v} + 2m\nabla v\nabla\tilde{F},\end{aligned}\tag{2.7}$$

where the last equality used

$$(m-1)\Delta v = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} = -\frac{1}{\alpha}\left(\tilde{F} + (\alpha-1)\frac{|\nabla v|^2}{v}\right).$$

Denote by $B_p(R)$ the geodesic ball centered at p with radius R . Take a cut-off function ϕ (see [12]) satisfying $\text{supp}(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$ and

$$\begin{aligned}\frac{|\nabla\phi|^2}{\phi} & \leq \frac{C}{R^2}, \\ -\Delta\phi & \leq \frac{C}{R^2}\left(1 + \sqrt{K}\coth(\sqrt{K}R)\right),\end{aligned}\tag{2.8}$$

where C is a constant depending only on n . Define $G = t\phi\tilde{F}$. Next we will apply maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then at the point (x_0, s) , it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla\tilde{F} = -\frac{\tilde{F}}{\phi}\nabla\phi$$

and by use of (2.7), we have

$$\begin{aligned}0 \leq \mathcal{L}(G) = & s\phi\mathcal{L}(\tilde{F}) - s(m-1)v\tilde{F}\Delta\phi - 2s(m-1)v\nabla\tilde{F}\nabla\phi + \phi\tilde{F} \\ = & s\phi\mathcal{L}(\tilde{F}) - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}\end{aligned}$$

$$\begin{aligned} &\leq s\phi \left(-\frac{1}{a\alpha^2}(\tilde{F} + (\alpha - 1)\frac{|\nabla v|^2}{v})^2 + 2(m-1)MK\frac{|\nabla v|^2}{v} - 2m\nabla v \frac{\nabla\phi}{\phi}\tilde{F} \right) \\ &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}. \end{aligned}$$

Let $\frac{|\nabla v|^2}{v} = \mu\tilde{F}$ at the point (x_0, s) . Then we have $\mu \geq 0$ and

$$\begin{aligned} 0 &\leq -\frac{1}{a\alpha^2}s\phi\tilde{F}^2(1 + (\alpha - 1)\mu)^2 + 2(m-1)MKs\phi\mu\tilde{F} - 2ms\phi\nabla v \frac{\nabla\phi}{\phi}\tilde{F} \\ &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\ &\leq -\frac{1}{a\alpha^2s\phi}(1 + (\alpha - 1)\mu)^2G^2 + 2(m-1)MK\mu G + 2mG\frac{|\nabla\phi|}{s^{\frac{1}{2}}\phi^{\frac{3}{2}}}M^{\frac{1}{2}}\mu^{\frac{1}{2}}G^{\frac{1}{2}} \\ &\quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}. \end{aligned} \tag{2.9}$$

Multiplying the both sides of (2.9) with $\frac{\phi}{G}$ yields

$$\begin{aligned} &\frac{1}{a\alpha^2s}(1 + (\alpha - 1)\mu)^2G - 2m\frac{|\nabla\phi|}{s^{\frac{1}{2}}\phi^{\frac{1}{2}}}M^{\frac{1}{2}}\mu^{\frac{1}{2}}G^{\frac{1}{2}} \\ &\leq 2(m-1)MK\mu\phi - (m-1)v\Delta\phi + 2(m-1)v\frac{|\nabla\phi|^2}{\phi} + \frac{\phi}{s} \\ &\leq 2(m-1)MK\mu - (m-1)M\Delta\phi + 2(m-1)M\frac{|\nabla\phi|^2}{\phi} + \frac{1}{s}. \end{aligned} \tag{2.10}$$

For the inequality $Ax^2 - 2Bx \leq C$, we have $x \leq \frac{2B}{A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}$. Applying this inequality to (2.10) by setting $x = G^{\frac{1}{2}}$ gives

$$\begin{aligned} G^{\frac{1}{2}} &\leq \frac{2a\alpha^2mM^{\frac{1}{2}}s^{\frac{1}{2}}\mu^{\frac{1}{2}}|\nabla\phi|}{(1 + (\alpha - 1)\mu)^2\phi^{\frac{1}{2}}} + \left\{ \frac{a\alpha^2s}{(1 + (\alpha - 1)\mu)^2} \left(\frac{1}{s} + 2(m-1)MK\mu \right. \right. \\ &\quad \left. \left. - (m-1)M\Delta\phi + 2(m-1)M\frac{|\nabla\phi|^2}{\phi} \right) \right\}^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

Obviously,

$$\begin{aligned} \frac{2a\alpha^2mM^{\frac{1}{2}}s^{\frac{1}{2}}\mu^{\frac{1}{2}}|\nabla\phi|}{(1 + (\alpha - 1)\mu)^2\phi^{\frac{1}{2}}} &= \frac{2a\alpha^2mM^{\frac{1}{2}}s^{\frac{1}{2}}((\alpha - 1)\mu)^{\frac{1}{2}}|\nabla\phi|}{(\alpha - 1)^{\frac{1}{2}}(1 + (\alpha - 1)\mu)^2\phi^{\frac{1}{2}}} \\ &\leq \frac{a\alpha^2mM^{\frac{1}{2}}s^{\frac{1}{2}}|\nabla\phi|}{(\alpha - 1)^{\frac{1}{2}}\phi^{\frac{1}{2}}} \end{aligned}$$

and

$$\begin{aligned} \frac{a\alpha^2s}{(1 + (\alpha - 1)\mu)^2} \left(\frac{1}{s} + 2(m-1)MK\mu \right) &\leq a\alpha^2 + \frac{2a\alpha^2s(m-1)MK}{\alpha - 1} \frac{(\alpha - 1)\mu}{(1 + (\alpha - 1)\mu)^2} \\ &\leq a\alpha^2 \left(1 + \frac{(m-1)MKs}{2(\alpha - 1)} \right). \end{aligned}$$

Thus, by use of (2.8), we have

$$\begin{aligned} G^{\frac{1}{2}}(x, T) &\leq G^{\frac{1}{2}}(x_0, s) \leq \frac{a\alpha^2 m M^{\frac{1}{2}} s^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \frac{|\nabla\phi|}{\phi^{\frac{1}{2}}} + \left\{ a\alpha^2 \left(1 + \frac{(m-1)MKs}{2(\alpha-1)} \right. \right. \\ &\quad \left. \left. - (m-1)Ms\Delta\phi + 2(m-1)Ms\frac{|\nabla\phi|^2}{\phi} \right) \right\}^{\frac{1}{2}} \\ &\leq \frac{a\alpha^2 m M^{\frac{1}{2}} T^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \frac{C}{R} + a^{\frac{1}{2}} T^{\frac{1}{2}} \alpha \left\{ \frac{1}{T} + \frac{(m-1)MK}{2(\alpha-1)} \right. \\ &\quad \left. + (m-1)M \frac{C}{R^2} \left(1 + \sqrt{K} \coth(\sqrt{K}R) \right) \right\}^{\frac{1}{2}}. \end{aligned}$$

Hence, for all $x \in B_p(R)$, it holds that

$$\begin{aligned} \tilde{F}^{\frac{1}{2}}(x, T) &\leq \frac{a\alpha^2 m M^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \frac{C}{R} + a^{\frac{1}{2}} \alpha \left\{ \frac{1}{T} + \frac{(m-1)MK}{2(\alpha-1)} \right. \\ &\quad \left. + (m-1)M \frac{C}{R^2} \left(1 + \sqrt{K} \coth(\sqrt{K}R) \right) \right\}^{\frac{1}{2}}. \end{aligned} \quad (2.12)$$

Since T is arbitrary, we obtain, for $x \in B_p(R)$

$$\begin{aligned} \tilde{F}^{\frac{1}{2}}(x, t) &\leq \frac{a\alpha^2 m M^{\frac{1}{2}}}{(\alpha-1)^{\frac{1}{2}}} \frac{C}{R} + a^{\frac{1}{2}} \alpha \left\{ \frac{1}{t} + \frac{(m-1)MK}{2(\alpha-1)} \right. \\ &\quad \left. + (m-1)M \frac{C}{R^2} \left(1 + \sqrt{K} \coth(\sqrt{K}R) \right) \right\}^{\frac{1}{2}}. \end{aligned}$$

We complete the proof of Theorem 1.1.

Proof of Corollary 1.2. Along the line of Li-Yau, we will establish Harnack inequality from a general estimate

$$\frac{|\nabla v|^2}{v} - \alpha(t) \frac{v_t}{v} - \varphi(t) \leq 0. \quad (2.13)$$

Rewrite (2.13) as

$$-\frac{v_t}{v} \leq \frac{1}{\alpha(t)} \left(\varphi(t) - \frac{|\nabla v|^2}{v} \right).$$

Let $f = \log v$. Then we have

$$\begin{aligned} -f_t &= -\frac{v_t}{v} \leq \frac{1}{\alpha(t)} \left(\varphi(t) - \frac{|\nabla v|^2}{v} \right) \\ &\leq \frac{1}{\alpha(t)} (\varphi(t) - \tilde{M} |\nabla f|^2). \end{aligned}$$

Let γ be a shortest geodesic joining x_1 and x_2 , $\gamma : [t_1, t_2] \rightarrow M^n$, $\gamma(t_1) = x_1$, $\gamma(t_2) = x_2$. We define a curve ζ in $M^n \times (0, \infty)$, $\zeta : [t_1, t_2] \rightarrow M^n \times (0, \infty)$ by $\zeta(t) = (\gamma(t), t)$. Then we have $\zeta(t_1) = (x_1, t_1)$, $\zeta(t_2) = (x_2, t_2)$. Denote by $\rho = d(x_1, x_2)$, then we have $|\dot{\gamma}| = \frac{\rho}{(t_2 - t_1)}$ and

$$f(x_1, t_1) - f(x_2, t_2) = \int_{t_2}^{t_1} \frac{d}{dt} f(\zeta(t)) dt$$

$$\begin{aligned}
&= \int_{t_2}^{t_1} (\langle \dot{\gamma}, \nabla f \rangle + f_t) dt \\
&= \int_{t_1}^{t_2} (-\langle \dot{\gamma}, \nabla f \rangle + (-f_t)) dt \\
&\leq \int_{t_1}^{t_2} \left(|\dot{\gamma}| |\nabla f| + \frac{1}{\alpha(t)} (\varphi(t) - \tilde{M} |\nabla f|^2) \right) dt \\
&= \int_{t_1}^{t_2} \left(-\frac{\tilde{M}}{\alpha(t)} |\nabla f|^2 + |\dot{\gamma}| |\nabla f| \right) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt \\
&\leq \frac{\rho^2}{4\tilde{M}(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha(t) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt, \tag{2.14}
\end{aligned}$$

where the last inequality used $-Ax^2 + Bx \leq \frac{B^2}{4A}$ and $|\dot{\gamma}| = \frac{\rho}{(t_2 - t_1)}$.

Let $\alpha > 1$ be a constant, $\varphi = \frac{\alpha^2}{2(\alpha-1)} a(m-1)MK + \frac{a\alpha^2}{t}$. We have from (2.14)

$$\begin{aligned}
f(x_1, t_1) - f(x_2, t_2) &\leq \int_{t_1}^{t_2} \left\{ \frac{\alpha \rho^2}{4\tilde{M}(t_2 - t_1)^2} + \left(\frac{\alpha}{2(\alpha-1)} a(m-1)MK + \frac{a\alpha}{t} \right) \right\} dt \\
&= \frac{\alpha \rho^2}{4\tilde{M}(t_2 - t_1)} + \frac{\alpha}{2(\alpha-1)} a(m-1)MK(t_2 - t_1) + a\alpha \log \frac{t_2}{t_1}. \tag{2.15}
\end{aligned}$$

Therefore, we arrive at

$$v(x_1, t_1) \leq v(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{a\alpha} \exp \left\{ \frac{\alpha \rho^2}{4\tilde{M}(t_2 - t_1)} + \frac{\alpha}{2(\alpha-1)} a(m-1)MK(t_2 - t_1) \right\}.$$

We complete the proof of Corollary 1.2.

3 Proof of Theorem 1.2

Define $\bar{F} = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v}$, where $\alpha = e^{2(m-1)MKt}$ is a function depending on t . Under the assumption that $\text{Ric} \geq -K$, we have from (2.2)

$$\begin{aligned}
\mathcal{L}(\bar{F}) &\leq -\frac{2}{n(m-1)} ((m-1)\Delta v)^2 + 2(m-1)MK \frac{|\nabla v|^2}{v} + 2m \nabla v \nabla \bar{F} \\
&\quad - ((m-1)\Delta v)^2 + (1-\alpha) \left(\frac{v_t}{v} \right)^2 - \alpha' \frac{v_t}{v} \\
&= -\frac{1}{a} ((m-1)\Delta v)^2 + 2(m-1)MK \frac{|\nabla v|^2}{v} + 2m \nabla v \nabla \bar{F} - \alpha' \frac{v_t}{v} \\
&\quad + (1-\alpha) \left(\frac{v_t}{v} \right)^2, \tag{3.1}
\end{aligned}$$

and hence

$$\begin{aligned}
\mathcal{L}(\alpha^{-1} \bar{F}) &= (\alpha^{-1})' \bar{F} + \alpha^{-1} \mathcal{L}(\bar{F}) \\
&\leq (\alpha^{-1})' \frac{|\nabla v|^2}{v} - \alpha (\alpha^{-1})' \frac{v_t}{v} \\
&\quad - \frac{1}{a} \alpha^{-1} ((m-1)\Delta v)^2 + 2(m-1)MK \alpha^{-1} \frac{|\nabla v|^2}{v} + 2m \alpha^{-1} \nabla v \nabla \bar{F} - \alpha' \alpha^{-1} \frac{v_t}{v}
\end{aligned}$$

$$\begin{aligned}
& + (1 - \alpha)\alpha^{-1} \left(\frac{v_t}{v} \right)^2 \\
& = -\frac{1}{a}\alpha^{-1}((m-1)\Delta v)^2 + 2m\alpha^{-1}\nabla v \nabla \bar{F} + (1 - \alpha)\alpha^{-1} \left(\frac{v_t}{v} \right)^2 \\
& \leq -\frac{1}{a}\alpha^{-1}((m-1)\Delta v)^2 + 2m\alpha^{-1}\nabla v \nabla \bar{F} \\
& = -\frac{1}{a}\alpha^{-1} \left((\alpha^{-1} - 1) \frac{|\nabla v|^2}{v} - \alpha^{-1}\bar{F} \right)^2 + 2m\nabla v \nabla (\alpha^{-1}\bar{F}), \tag{3.2}
\end{aligned}$$

where we used $(m-1)\Delta v = (\alpha^{-1} - 1) \frac{|\nabla v|^2}{v} - \alpha^{-1}\bar{F}$.

Recall that one can construct a cut-off function ϕ as before, which satisfies $\text{supp}(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$ and

$$\begin{aligned}
\frac{|\nabla \phi|^2}{\phi} & \leq \frac{C}{R^2}, \\
-\Delta \phi & \leq \frac{C}{R^2} \left(1 + \sqrt{K}R \coth(\sqrt{K}R) \right),
\end{aligned}$$

where C is a constant depending only on n . Define $G = t\phi\alpha^{-1}\bar{F}$. Next we are to apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$, and assume that $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then at the point (x_0, s) , it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla(\alpha^{-1}\bar{F}) = -\frac{\alpha^{-1}\bar{F}}{\phi} \nabla \phi$$

and

$$\begin{aligned}
0 \leq \mathcal{L}(G) & = s\phi\mathcal{L}(\alpha^{-1}\bar{F}) - s(m-1)v\alpha^{-1}\bar{F}\Delta\phi - 2s(m-1)v\nabla(\alpha^{-1}\bar{F})\nabla\phi + \phi\alpha^{-1}\bar{F} \\
& = s\phi\mathcal{L}(\alpha^{-1}\bar{F}) - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
& \leq -\frac{s\phi\alpha^{-1}}{a} \left((\alpha^{-1} - 1) \frac{|\nabla v|^2}{v} - \alpha^{-1}\bar{F} \right)^2 + 2ms\phi\nabla v \nabla(\alpha^{-1}\bar{F}) \\
& \quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
& \leq -\frac{s\phi\alpha^{-1}}{a}(\alpha^{-1} - 1)^2 \frac{|\nabla v|^4}{v^2} - \frac{\alpha^{-1}}{as\phi}G^2 + \frac{2\alpha^{-1}(\alpha^{-1} - 1)}{a} \frac{|\nabla v|^2}{v}G - 2m\nabla v \frac{\nabla\phi}{\phi}G \\
& \quad - (m-1)v\frac{\Delta\phi}{\phi}G + 2(m-1)v\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s} \\
& \leq -\frac{\alpha^{-1}}{as\phi}G^2 + \frac{2\alpha^{-1}(\alpha^{-1} - 1)}{a} \frac{|\nabla v|^2}{v}G + 2mM^{\frac{1}{2}} \frac{|\nabla\phi|}{\phi} \frac{|\nabla v|}{v^{\frac{1}{2}}}G \\
& \quad - (m-1)M\frac{\Delta\phi}{\phi}G + 2(m-1)M\frac{|\nabla\phi|^2}{\phi^2}G + \frac{G}{s}. \tag{3.3}
\end{aligned}$$

Multiplying the both sides of (3.3) with $\frac{\alpha as\phi}{G}$ yields

$$G(x, T) \leq G(x_0, s) \leq -2(1 - \alpha^{-1})s\phi \frac{|\nabla v|^2}{v} + 2mM^{\frac{1}{2}}as\alpha|\nabla\phi| \frac{|\nabla v|}{v^{\frac{1}{2}}} - (m-1)Mas\alpha\Delta\phi$$

$$\begin{aligned}
& + 2(m-1)Mas\alpha \frac{|\nabla\phi|^2}{\phi} + a\alpha\phi \\
& \leq \frac{m^2Ma^2\alpha^2s}{2(1-\alpha^{-1})} \frac{|\nabla\phi|^2}{\phi} - (m-1)Mas\alpha\Delta\phi + 2(m-1)Mas\alpha \frac{|\nabla\phi|^2}{\phi} + a\alpha\phi \\
& \leq \left(\frac{m^2Ma^2\alpha^2T}{2(1-\alpha^{-1})} + 3(m-1)MaT\alpha \right) \frac{C}{R^2} \\
& \quad + (m-1)MaT\alpha\sqrt{K} \coth(\sqrt{K}R) \frac{C}{R} + a\alpha.
\end{aligned} \tag{3.4}$$

Hence, for all $x \in B_p(R)$, it holds that

$$\begin{aligned}
(\alpha^{-1}\overline{F})(x, T) & \leq \left(\frac{m^2Ma^2\alpha^2}{2(1-\alpha^{-1})} + 3(m-1)Ma\alpha \right) \frac{C}{R^2} \\
& \quad + (m-1)Ma\alpha\sqrt{K} \coth(\sqrt{K}R) \frac{C}{R} + \frac{a\alpha}{T},
\end{aligned}$$

and hence,

$$\begin{aligned}
\overline{F}(x, T) & \leq \left(\frac{m^2Ma^2\alpha^3}{2(1-\alpha^{-1})} + 3(m-1)Ma\alpha^2 \right) \frac{C}{R^2} \\
& \quad + (m-1)Ma\alpha^2\sqrt{K} \coth(\sqrt{K}R) \frac{C}{R} + \frac{a\alpha^2}{T},
\end{aligned}$$

Since T is arbitrary, we complete the proof of Theorem 1.2.

Proof of Corollary 1.4. Choosing $\alpha(t) = e^{2(m-1)MKt}$, $\varphi(t) = \frac{a\alpha^2(t)}{t}$ in (2.14), we get

$$\begin{aligned}
\log v(x_1, t_1) - \log v(x_2, t_2) & \leq \int_{t_1}^{t_2} \left(\frac{\rho^2\alpha}{4\tilde{M}(t_2-t_1)^2} + \frac{a\alpha}{t} \right) dt \\
& \leq \int_{t_1}^{t_2} \left(\frac{\rho^2\alpha}{4\tilde{M}(t_2-t_1)^2} + \frac{a\alpha}{t_1} \right) dt \\
& = \left(\frac{\rho^2}{4\tilde{M}(t_2-t_1)^2} + \frac{a}{t_1} \right) \frac{e^{2(m-1)MKt_2} - e^{2(m-1)MKt_1}}{2(m-1)MK},
\end{aligned} \tag{3.5}$$

which concludes the proof of Corollary 1.4.

4 Proof of Theorem 1.3

Under the assumption that $\text{Ric} \geq -K$, we have from (2.2)

$$\begin{aligned}
\mathcal{L}(F) & \leq -\frac{1}{a}((m-1)\Delta v)^2 + 2(m-1)MK \frac{|\nabla v|^2}{v} + 2m\nabla v \nabla F - \alpha' \frac{v_t}{v} \\
& \quad - \varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2 \\
& = -\frac{1}{a}((m-1)\Delta v + \varphi)^2 + \frac{2}{a}\varphi((m-1)\Delta v) + \frac{1}{a}\varphi^2 + 2(m-1)MK \frac{|\nabla v|^2}{v} \\
& \quad + 2m\nabla v \nabla F - \alpha' \frac{v_t}{v} - \varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{a}((m-1)\Delta v + \varphi)^2 + \frac{2}{a}\varphi \left(\frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right) + \frac{1}{a}\varphi^2 + 2(m-1)MK \frac{|\nabla v|^2}{v} \\
&\quad + 2m\nabla v \nabla F - \alpha' \frac{v_t}{v} - \varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2 \\
&= -\frac{1}{a}((m-1)\Delta v + \varphi)^2 + 2m\nabla v \nabla F \\
&\quad - \left(\frac{2}{a}\varphi - 2(m-1)MK \right) \left(\frac{|\nabla v|^2}{v} - \frac{\frac{2}{a}\varphi - \alpha'}{\frac{2}{a}\varphi - 2(m-1)MK} \frac{v_t}{v} - \varphi \right) \\
&\quad - \left(\frac{2}{a}\varphi - 2(m-1)MK \right) \varphi + \frac{1}{a}\varphi^2 - \varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2. \tag{4.1}
\end{aligned}$$

Take

$$\begin{aligned}
\varphi(t) &= a(m-1)MK \{ \coth((m-1)MKt) + 1 \} \\
\alpha(t) &= 1 + \frac{\cosh((m-1)MKt) \sinh((m-1)MKt) - (m-1)MKt}{\sinh^2((m-1)MKt)}, \tag{4.2}
\end{aligned}$$

then $\alpha(t)$ and $\varphi(t)$ satisfy the following equations:

$$\begin{cases} -\left(\frac{2}{a}\varphi - 2(m-1)MK\right) \varphi + \frac{1}{a}\varphi^2 - \varphi' = 0 \\ \frac{\frac{2}{a}\varphi - \alpha'}{\frac{2}{a}\varphi - 2(m-1)MK} = \alpha. \end{cases} \tag{4.3}$$

Moreover, it is easy to see that $\alpha \geq 1$. Putting (4.3) into (4.1), we obtain

$$\begin{aligned}
\mathcal{L}(F) &\leq -\frac{1}{a}((m-1)\Delta v + \varphi)^2 + 2m\nabla v \nabla F - 2(m-1)MK \coth((m-1)MKt)F \\
&= -\frac{1}{a\alpha^2} \left(F + (\alpha-1) \left(\frac{|\nabla v|^2}{v} - \varphi \right) \right)^2 + 2m\nabla v \nabla F \\
&\quad - 2(m-1)MK \coth((m-1)MKt)F, \tag{4.4}
\end{aligned}$$

where we used

$$(m-1)\Delta v + \varphi = \frac{v_t}{v} - \frac{|\nabla v|^2}{v} + \varphi = -\frac{1}{\alpha} \left(F + (\alpha-1) \left(\frac{|\nabla v|^2}{v} - \varphi \right) \right).$$

We can construct a cut-off function ϕ as before, satisfying $\text{supp}(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$ and

$$\begin{aligned}
\frac{|\nabla \phi|^2}{\phi} &\leq \frac{C}{R^2}, \\
-\Delta \phi &\leq \frac{C}{R^2} \left(1 + \sqrt{K}R \coth(\sqrt{K}R) \right),
\end{aligned}$$

where C is a constant which depends only on n . Define $G = \beta(t)\phi F$, where $\beta(t)$ is a positive function to be determined. Next we are to apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$, and assume that $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then at the point (x_0, s) , it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla F = -\frac{F}{\phi} \nabla \phi$$

and

$$\begin{aligned}
0 &\leq \mathcal{L}(G) = \beta' \phi F + \beta \phi \mathcal{L}(F) - (m-1)\beta v F \Delta \phi - 2(m-1)\beta v \nabla \phi \nabla F \\
&= \frac{\beta'}{\beta} G + \beta \phi \mathcal{L}(F) - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G \\
&\leq \frac{\beta'}{\beta} G - \frac{\beta \phi}{a\alpha^2} \left(F + (\alpha-1) \left(\frac{|\nabla v|^2}{v} - \varphi \right) \right)^2 + 2m\beta \phi \nabla v \nabla F \\
&\quad - 2(m-1)\beta \phi MK \coth((m-1)MKs) F - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G \\
&= \frac{\beta'}{\beta} G - \frac{1}{a\alpha^2 \beta \phi} G^2 - \frac{\beta \phi (\alpha-1)^2}{a\alpha^2} \left(\frac{|\nabla v|^2}{v} - \varphi \right)^2 - 2 \frac{(\alpha-1)}{a\alpha^2} \left(\frac{|\nabla v|^2}{v} - \varphi \right) G - 2m \nabla v \frac{\nabla \phi}{\phi} G \\
&\quad - 2(m-1)MK \coth((m-1)MKs) G - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G \\
&\leq \frac{\beta'}{\beta} G - \frac{1}{a\alpha^2 \beta \phi} G^2 - 2 \frac{(\alpha-1)}{a\alpha^2} \left(\frac{|\nabla v|^2}{v} - \varphi \right) G - 2m \nabla v \frac{\nabla \phi}{\phi} G \\
&\quad - 2(m-1)MK \coth((m-1)MKs) G - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G. \quad (4.5)
\end{aligned}$$

Multiplying the both sides of (4.5) with $\frac{a\alpha^2 \beta \phi}{G}$ yields

$$\begin{aligned}
G(x_0, s) &\leq a\alpha^2 \beta \left\{ \frac{\beta'}{\beta} \phi - 2 \frac{(\alpha-1)}{a\alpha^2} \left(\frac{|\nabla v|^2}{v} - \varphi \right) \phi - 2m \nabla v \nabla \phi \right. \\
&\quad \left. - 2(m-1)MK \coth((m-1)MKs) \phi - (m-1)v \Delta \phi \right. \\
&\quad \left. + 2(m-1)v \frac{|\nabla \phi|^2}{\phi} \right\} \\
&= a\alpha^2 \beta \left\{ \left(\frac{\beta'}{\beta} + 2 \frac{(\alpha-1)}{a\alpha^2} \varphi - 2(m-1)MK \coth((m-1)MKs) \right) \phi \right. \\
&\quad \left. - (m-1)v \Delta \phi + 2(m-1)v \frac{|\nabla \phi|^2}{\phi} \right. \\
&\quad \left. - 2 \frac{(\alpha-1)\phi}{a\alpha^2} \frac{|\nabla v|^2}{v} + 2mv^{\frac{1}{2}} |\nabla \phi| \frac{|\nabla v|}{v^{\frac{1}{2}}} \right\} \\
&\leq \beta \left\{ 2(\alpha-1)\varphi - a\alpha^2 \left(2(m-1)MK \coth((m-1)MKs) - \frac{\beta'}{\beta} \right) \right\} \phi \\
&\quad + a\alpha^2 \beta \left\{ -(m-1)\Delta \phi + 2(m-1) \frac{|\nabla \phi|^2}{\phi} + \frac{am^2\alpha^2}{2(\alpha-1)} \frac{|\nabla \phi|^2}{\phi} \right\} M, \quad (4.6)
\end{aligned}$$

where the last inequality used $-Ax^2 + Bx \leq \frac{B^2}{4A}$. Choosing $\beta(t) = \tanh((m-1)MKt)$, we have $\frac{\beta'}{\beta} = \frac{(m-1)MK}{\sinh((m-1)MKt) \cosh((m-1)MKt)}$ and

$$2(\alpha-1)\varphi - a\alpha^2 \left(2(m-1)MK \coth((m-1)MKt) - \frac{\beta'}{\beta} \right) \leq 0.$$

Note that $\alpha, \frac{\beta}{\alpha-1}$ is bounded uniformly and β is non-decreasing. Thus from (4.6) we obtain

$$G(x, T) \leq G(x_0, s)$$

$$\begin{aligned}
&\leq a\alpha^2\beta \left\{ -(m-1)\Delta\phi + 2(m-1)\frac{|\nabla\phi|^2}{\phi} + \frac{am^2\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi} \right\} M \\
&\leq \left\{ a(m-1)\beta(T) \left(\frac{C}{R^2} + \frac{C\sqrt{K}\coth(\sqrt{K}R)}{R} \right) + a^2m^2\frac{C}{R^2} \right\} M. \quad (4.7)
\end{aligned}$$

Hence, for all $x \in B_p(R)$, one has

$$F(x, T) \leq \left\{ a(m-1) \left(\frac{C}{R^2} + \frac{C\sqrt{K}\coth(\sqrt{K}R)}{R} \right) + a^2m^2\frac{C}{R^2\beta(T)} \right\} M.$$

Since T is arbitrary, we obtain

$$\begin{aligned}
\frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} - \varphi(t) &\leq \left\{ a(m-1) \left(\frac{C}{R^2} + \frac{C\sqrt{K}\coth(\sqrt{K}R)}{R} \right) \right. \\
&\quad \left. + a^2m^2\frac{C}{R^2\tanh((m-1)MKt)} \right\} M. \quad (4.8)
\end{aligned}$$

We complete the proof of Theorem 1.3.

Proof of Corollary 1.6. Putting $\alpha(t), \varphi(t)$ given by (1.18) into (2.14) gives

$$\begin{aligned}
&\log v(x_1, t_1) - \log v(x_2, t_2) \\
&\leq \frac{\rho^2}{4\tilde{M}(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha(t)dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)}dt \\
&= \left\{ \frac{\rho^2}{4\tilde{M}(t_2 - t_1)^2} \left(t + \frac{(m-1)MKt\coth((m-1)MKt) - 1}{(m-1)MK} \right) \right. \\
&\quad \left. + \frac{a}{2} \log \frac{\sinh(2(m-1)MKt) + \cosh(2(m-1)MKt) - 2(m-1)MKt - 1}{2(m-1)MK} \right\} \Big|_{t_1}^{t_2} \\
&= \frac{\rho^2}{4\tilde{M}(t_2 - t_1)} (1 + A_2(t_1, t_2)) + \log A_1(t_1, t_2),
\end{aligned}$$

where

$$\begin{aligned}
A_1(t_1, t_2) &= \left(\frac{\exp(2(m-1)MKt_2) - 2(m-1)MKt_2 - 1}{\exp(2(m-1)MKt_1) - 2(m-1)MKt_1 - 1} \right)^{\frac{a}{2}} \\
A_2(t_1, t_2) &= \frac{t_2\coth((m-1)MKt_2) - t_1\coth((m-1)MKt_1)}{t_2 - t_1}.
\end{aligned}$$

Therefore, we arrive at

$$v(x_1, t_1) \leq v(x_2, t_2) A_1(t_1, t_2) \exp \left\{ \frac{\rho^2}{4\tilde{M}(t_2 - t_1)} (1 + A_2(t_1, t_2)) \right\}.$$

We complete the proof of Corollary 1.6.

5 Proof of Theorem 1.4

The proof of Theorem 1.4 is similar to the proof of Theorem 1.3. Under the assumption that $\text{Ric} \geq -K$, we have from (2.2)

$$\mathcal{L}(F) \leq -\frac{1}{a}((m-1)\Delta v)^2 + 2(m-1)MK\frac{|\nabla v|^2}{v} + 2m\nabla v \nabla F - \alpha' \frac{v_t}{v}$$

$$\begin{aligned}
& -\varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2 \\
& = -\frac{1}{a} \left((m-1)\Delta v + a \left(\frac{1}{t} + (m-1)MK \right) \right)^2 + 2(m-1)\Delta v \left(\frac{1}{t} + (m-1)MK \right) \\
& \quad + a \left(\frac{1}{t} + (m-1)MK \right)^2 + 2(m-1)MK \frac{|\nabla v|^2}{v} + 2m\nabla v \nabla F - \alpha' \frac{v_t}{v} \\
& \quad - \varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2 \\
& = -\frac{1}{a} \left((m-1)\Delta v + a \left(\frac{1}{t} + (m-1)MK \right) \right)^2 - \frac{2}{t} \left\{ \frac{|\nabla v|^2}{v} \right. \\
& \quad \left. - \frac{2(\frac{1}{t} + (m-1)MK) - \alpha' v_t}{\frac{2}{t}} \frac{v_t}{v} - \varphi \right\} - \frac{2}{t} \varphi + a \left(\frac{1}{t} + (m-1)MK \right)^2 \\
& \quad - \varphi' + (1-\alpha) \left(\frac{v_t}{v} \right)^2 + 2m\nabla v \nabla F
\end{aligned} \tag{5.1}$$

Taking

$$\begin{aligned}
\varphi(t) &= \frac{a}{t} + a(m-1)MK + \frac{a}{3}((m-1)MK)^2 t, \\
\alpha(t) &= 1 + \frac{2}{3}(m-1)MKt,
\end{aligned} \tag{5.2}$$

then $\alpha(t)$ and $\varphi(t)$ satisfy the following equations:

$$\begin{cases} -\frac{2}{t}\varphi + a \left(\frac{1}{t} + (m-1)MK \right)^2 - \varphi' = 0 \\ \frac{2(\frac{1}{t} + (m-1)MK) - \alpha'}{\frac{2}{t}} = \alpha. \end{cases} \tag{5.3}$$

Moreover, it is easy to see that $\alpha \geq 1$. Putting (5.3) into (5.1), we obtain

$$\begin{aligned}
\mathcal{L}(F) &\leq -\frac{1}{a} \left((m-1)\Delta v + a \left(\frac{1}{t} + (m-1)MK \right) \right)^2 - \frac{2}{t}F + 2m\nabla v \nabla F \\
&= -\frac{1}{a\alpha^2} \left\{ F + (\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 t \right) \right\}^2 \\
&\quad - \frac{2}{t}F + 2m\nabla v \nabla F,
\end{aligned} \tag{5.4}$$

where we used

$$\begin{aligned}
& (m-1)\Delta v + a \left(\frac{1}{t} + (m-1)MK \right) \\
&= -\frac{1}{\alpha} \left\{ F + (\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 t \right) \right\}.
\end{aligned}$$

Construct a cut-off function ϕ as before, which satisfies $\text{supp}(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$ and

$$\begin{aligned}
& \frac{|\nabla \phi|^2}{\phi} \leq \frac{C}{R^2}, \\
& -\Delta \phi \leq \frac{C}{R^2} \left(1 + \sqrt{K}R \coth(\sqrt{K}R) \right),
\end{aligned}$$

where C is a constant depending only on n . Define $G = \beta(t)\phi F$, where $\beta(t)$ is a positive function to be determined. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$, and assume that $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then at the point (x_0, s) , it holds that

$$\mathcal{L}(G) \geq 0, \quad \nabla F = -\frac{F}{\phi} \nabla \phi$$

and

$$\begin{aligned} 0 \leq \mathcal{L}(G) &= \beta' \phi F + \beta \phi \mathcal{L}(F) - (m-1)\beta v F \Delta \phi - 2(m-1)\beta v \nabla \phi \nabla F \\ &= \frac{\beta'}{\beta} G + \beta \phi \mathcal{L}(F) - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G \\ &\leq \frac{\beta'}{\beta} G - \frac{\beta \phi}{a\alpha^2} \left\{ F + (\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 s \right) \right\}^2 \\ &\quad - \frac{2}{s} \beta \phi F + 2m\beta \phi \nabla v \nabla F - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G \\ &= \frac{\beta'}{\beta} G - \frac{G^2}{a\alpha^2 \beta \phi} - \frac{2G}{a\alpha^2} \left\{ (\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 s \right) \right\} \\ &\quad - \frac{\beta \phi}{a\alpha^2} \left\{ (\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 s \right) \right\}^2 \\ &\quad - \frac{2}{s} G - 2m \nabla v \frac{\nabla \phi}{\phi} G - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G \\ &\leq \frac{\beta'}{\beta} G - \frac{G^2}{a\alpha^2 \beta \phi} - \frac{2G}{a\alpha^2} \left\{ (\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 s \right) \right\} \\ &\quad - \frac{2}{s} G - 2m \nabla v \frac{\nabla \phi}{\phi} G - (m-1)v \frac{\Delta \phi}{\phi} G + 2(m-1)v \frac{|\nabla \phi|^2}{\phi^2} G. \end{aligned} \quad (5.5)$$

Multiplying both sides of (5.5) with $\frac{a\alpha^2 \beta \phi}{G}$ yields

$$\begin{aligned} G(x_0, s) &\leq a\alpha^2 \beta \left\{ \frac{\beta'}{\beta} \phi - \frac{2}{a\alpha^2} \left((\alpha-1) \frac{|\nabla v|^2}{v} - \left(\frac{2a}{3}(m-1)MK + \frac{a}{3}((m-1)MK)^2 s \right) \right) \right\} \phi \\ &\quad - \frac{2}{s} \phi - 2m \nabla v \nabla \phi - (m-1)v \Delta \phi + 2(m-1)v \frac{|\nabla \phi|^2}{\phi} \Big\} \\ &= a\alpha^2 \beta \left\{ \left(\frac{\beta'}{\beta} + \frac{2}{3\alpha^2} \left(2(m-1)MK + ((m-1)MK)^2 s \right) - \frac{2}{s} \right) \phi \right. \\ &\quad \left. - (m-1)v \Delta \phi + 2(m-1)v \frac{|\nabla \phi|^2}{\phi} - 2 \frac{(\alpha-1)\phi}{a\alpha^2} \frac{|\nabla v|^2}{v} + 2mv^{\frac{1}{2}} |\nabla \phi| \frac{|\nabla v|}{v^{\frac{1}{2}}} \right\} \\ &\leq \frac{a\beta}{s} \left\{ \frac{2}{3} \left(2(m-1)MKs + ((m-1)MK)^2 s^2 \right) + s\alpha^2 \left(\frac{\beta'}{\beta} - \frac{2}{s} \right) \right\} \phi \\ &\quad + a\alpha^2 \beta \left\{ -(m-1)\Delta \phi + 2(m-1) \frac{|\nabla \phi|^2}{\phi} + \frac{am^2 \alpha^2}{2(\alpha-1)} \frac{|\nabla \phi|^2}{\phi} \right\} M. \end{aligned} \quad (5.6)$$

Choose $\beta(t) = \tanh((m-1)MKt)$, then we have $\frac{\beta'}{\beta} = \frac{(m-1)MK}{\sinh((m-1)MKt) \cosh((m-1)MKt)}$ and

$$\frac{2}{3} \left(2(m-1)MKt + ((m-1)MK)^2 t^2 \right) + t\alpha^2 \left(\frac{\beta'}{\beta} - \frac{2}{t} \right) \leq 0.$$

Note that $\frac{\beta}{\alpha-1}$ is bounded uniformly and α, β is non-decreasing. Thus from (5.6) we obtain

$$\begin{aligned} G(x, T) &\leq G(x_0, s) \\ &\leq a\alpha^2\beta \left\{ -(m-1)\Delta\phi + 2(m-1)\frac{|\nabla\phi|^2}{\phi} + \frac{am^2\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi} \right\} M \\ &\leq \left\{ a(m-1)\alpha^2(T)\beta(T) \left(\frac{C}{R^2} + \frac{C\sqrt{K}\coth(\sqrt{K}R)}{R} \right) + a^2m^2\alpha^4(T)\frac{C}{R^2} \right\} M. \end{aligned}$$

Hence, for $x \in B_p(R)$, we have

$$F(x, T) \leq \left\{ a(m-1)\alpha^2(T) \left(\frac{C}{R^2} + \frac{C\sqrt{K}\coth(\sqrt{K}R)}{R} \right) + \frac{a^2m^2\alpha^4(T)}{\beta(T)}\frac{C}{R^2} \right\} M.$$

Since T is arbitrary, we obtain

$$\begin{aligned} \frac{|\nabla v|^2}{v} - \alpha(t)\frac{v_t}{v} - \varphi(t) &\leq \left\{ a(m-1)\alpha^2(t) \left(\frac{C}{R^2} + \frac{C\sqrt{K}\coth(\sqrt{K}R)}{R} \right) \right. \\ &\quad \left. + \frac{a^2m^2\alpha^4(t)}{\beta(t)}\frac{C}{R^2} \right\} M. \end{aligned} \quad (5.7)$$

It completes the proof of Theorem 1.4.

Proof of Corollary 1.8. Recall the estimate (1.23), which implies

$$-\frac{v_t}{v} \leq \frac{1}{\alpha(t)} \left(\varphi(t) - \frac{|\nabla v|^2}{v} \right),$$

where $\alpha(t)$ and $\varphi(t)$ are given by (1.22). It follows from (2.14) that

$$\log v(x_1, t_1) - \log v(x_2, t_2) \leq \frac{\rho^2}{4\tilde{M}(t_2 - t_1)^2} \int_{t_1}^{t_2} \alpha(t) dt + \int_{t_1}^{t_2} \frac{\varphi(t)}{\alpha(t)} dt. \quad (5.8)$$

Choosing $\alpha(t) = 1 + \frac{2}{3}(m-1)MKt$ and $\varphi(t) = \frac{a}{t} + a(m-1)MK + \frac{a}{3}((m-1)MK)^2t$ in (5.8) concludes the proof of Corollary 1.8.

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